SEQUENCE AND SERIES

Sequences

The Concept of Sequence

Explain the concept of sequence

A Sequence is the arrangement of numbers or is a list of numbers following a clear pattern such that one number and the next are separated by comma (,).

Example: a_1, a_2, a_3, a_4 ……………………..

NB: Each number found in a Series or Sequence is called a ***term***.

Example 1

Find the next three terms in the following sequences.

a. 5, 8, 11, 14, 17,………………………………

b. 3, 7, 6, 10, 9, …………………………………

c. 1, 2, 4, 7, ………………………………………

d. 2, 9, 20, 35, …………………………………

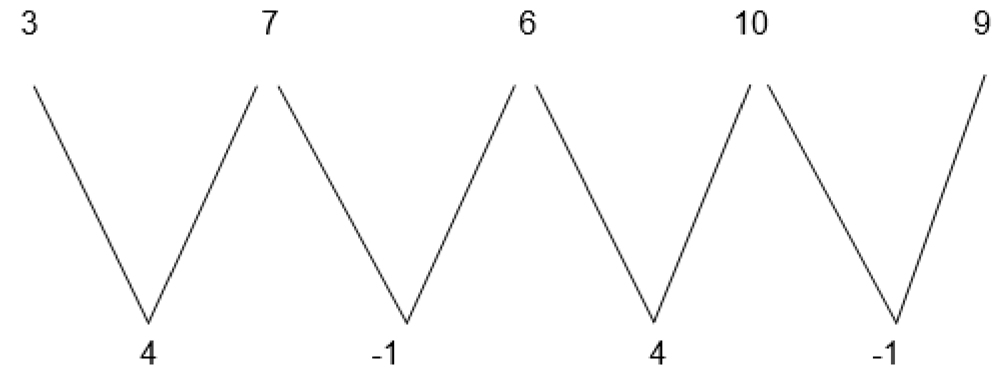
Solution:

(a)You can see that each term is less to the next by 3.

So next three terms are (17+3),(17+3+3) and 17+3+3x3)

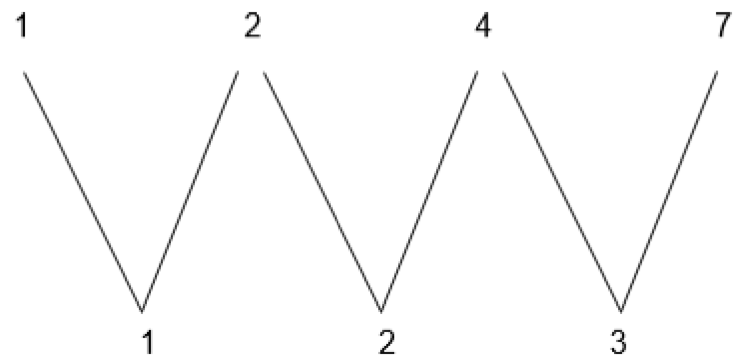
Which are 20, 23, and 26

(b)

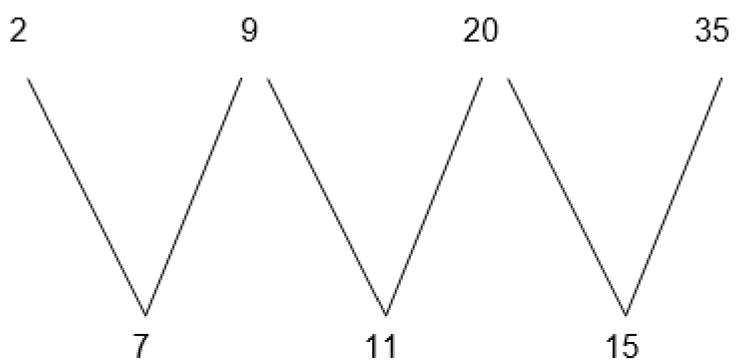


Alternately add 4 and subs tract 1. The sequence then extends to 13, 12, 16

(c)



We see that the difference is increasing by 1 each time. So the next three terms are 11, 16 and 22.



The differences are increased by 4 each time, so the next three terms are 54, 77 and 104.

Example 2

Write down the first three terms in the sequences where the n^{th} term is given by the formulae.

(a) 4n-3 (b) $\frac{n^2+1}{2}$ (c) $\frac{2}{n+1}$

Solution:

(a) 4n-3.

n=1, 4n-3 = 4×1-3=1

n=2, 4n-3 = 4×2-3=5

n=3, 4n-3 = 4× 3-3=9

∴The Sequence is 1, 5, 9, …………………

(b) $\frac{n^2+1}{2}$

n=1, $\frac{n^2+1}{2} = \frac{1^2+1}{2} = 1$

n=2, $\frac{n^2+1}{2} = \frac{2^2+1}{2} = \frac{5}{2}$

n=3, $\frac{n^2+1}{2} = \frac{3^2+1}{2} = 5$

∴ The sequence is 1, $\frac{5}{2}$, 5 ,……………

(c) $\frac{2}{n+1}$

n=1, $\frac{2}{n+1} = \frac{2}{1+1} = 1$

n=2, $\frac{2}{n+1} = \frac{2}{2+1} = \frac{2}{3}$

n=3, $\frac{2}{n+1} = \frac{2}{3+1} = \frac{1}{2}$

$\therefore$ The sequence is 1, $\frac{2}{3}$, $\frac{1}{2}$, ...

Example 3

The k^{th} term of a series is $k^2 + 4$

Find the sum of the first four terms in the series

Solution:

k=1, $k^2+4=1^2+4=5$

k=2, $k^2+4=2^2+4=8$

k=3, $k^2+4=3^2+4=13$

k=4, $k^2+4=4^2+4=20$

So the series is 5+8+13+20 and its sum is **46**

Example 4

Find the n^{th} term of the following sequences:

(a) 2, 3, 4, 5, 6, ...

(b) $\frac{1}{2}, \frac{2}{3}, \frac{3}{4}, \frac{4}{5},$ ....

Solution:

(a) First term = 1+1=2

Second term = 2+1=3

Third term = 3+1=4

Fourth term = 4+1=5 and so on

So the n^{th} is n+1.

(b) The numerators of fractions are 1, 2, 3, 4 .... and the fractions are 2, 3, 4, 5 ....

So the n^{th} term for numerator is n and that of denominator is n+1,

Combining them gives the n^{th} term of the fractions.

So the n^{th} term is $\frac{\text{numerator nth term}}{\text{Denominator nth term}}$ which is $\frac{n}{n+1}$

$\therefore$ The n^{th} term is $\frac{n}{n+1}$

Exercise 1

1. Write down the next three terms in the following sequences

(a) 1,5,9,13,17, ……………………

(b) 27, 24, 21,18, ……………………

(c) 5, 8, 9, 12, 13, ……………………

(d) 1, 8, 27, 64, ……………………

(e) 1, $\frac{-1}{2}$, $\frac{1}{4}$, $\frac{-1}{8}$, ……………………

2. Find the first three terms in the sequence:

a. $5n+2$

b. $1-3k$

c. n^2+n+1

d. 2^n

3. Find the sum of the first four terms of the series where the k^{th} term is given by:

a. $5k+3$

b. k^3-1

c. 2^k

4. Find the n^{th} term of these sequences:

(a) 2, 4, 5, 6, ………………………………

(b) 4, 5, 6, 7, ………………………………

(c) 10, 20, 30, 40,………………………………

(d) 2, 4, 6, 8,………………………………

(e) $1\frac{1}{2}$, $2\frac{1}{2}$, $3\frac{1}{2}$, $4\frac{1}{2}$, ……………………………

(f) $2\frac{1}{2}$, $3\frac{1}{3}$, $4\frac{1}{4}$, $5\frac{1}{5}$, ……………………………

An Arithmetic Progression (AP) and Geometric Progression (GP)

Identify an arithmetic progression (AP) and geometric progression (GP)

When the series or sequence is such that between two consecutive terms there is a difference which is fixed, then the series or sequence is called an arithmetic progression (A.P)

The fixed difference (number) between two consecutive terms is called the common difference (d)

Example 5

In the sequence 4, 7, 19, 13, 16 there is a common difference which is
7-4=10-7=13-10=16-13=3.

So the common difference (d)=3.

Note that in arithmetic progression (A.P) the difference between two successive terms is always the same.

Sometimes numbers may be decreasing instead of increasing, the arithmetic sequence or series while terms decrease have a negative number as a common difference.

Example 6

The common difference of the sequence 6, 4, 0, -2, …………………… is
4-6=2-4=0-2=-2-0=-2

So the common difference is **-2.**

In general if A_1, A_2, A_3, A_4, ……………………… A_n are the terms of the arithmetic sequence , then the common difference is ;

$$d=A_2-A_1=A_3-A_2=A_4-A_3=\ldots\ldots\ldots$$

Example 7

For each of the following sequences, find the common difference and write the next two terms.

(a) -2, 5, 12, 19, 26, ……………………………………

(b) $4\frac{1}{2}$, $3\frac{1}{4}$, $2\frac{3}{4}$, …..

Solution:

(a) d =5-(-2)=12-5=19-12   d=7

∴The next term to 26 is 26+7=33 and the next to 33 is 33+7=40.

So the next two terms are 33 and 40.

$d = 3\frac{1}{4} - 4\frac{1}{2} = 2 - 3\frac{3}{4},$

or $d = \frac{3}{4} - 2$

So $d= -\frac{5}{4},$

∴The common difference is $-\frac{5}{4},$

The next two terms are: $\left(\frac{3}{4} + \left(\frac{-5}{4}\right)\right)$and $\left(\frac{3}{4} + \left(\frac{-5}{4}\right)\right)+\left(\frac{-5}{4}\right)\right)$

Which are $\frac{-1}{2}$ and $\frac{-7}{4}$

Exercise 2

1. Find the common difference for each of the following sequence:

a. 11, 14, 17, 20, …………………………………

b. 2, 4, 6, 8, 10, ……………………………………

c. 0.1, 0.11, 0.111, 0.1111 , …… … … … … …

d. y, y+3, y+6, y+9, y+12, … …… … … … ……

2. State whetherthe following sequence are arithmetic or not:

a. 2, 5, 8, 11, 14, …………… ……………… ……

b. 1, 3, 4, 6, 7, 9, 10, ………………………………

c. y, y + x, y+2x, y+3x, … ………… ……

3. The temperatureat a mid day is 3^0c, and it falls by 2^0c each hour. Find the temperature at the end of the next four hours.

Geometric Progression (G.P).

When the series or Sequence is such that between two consecutive terms there is a ration which is fixed, then the series or sequence is called a geometric progression (G.P)

The fixed ratio(number) between two successive terms is called the common ratio (r).

Example 8

In 2, 4, 8, 16, 32, … … …… … … …….
There is a common ration which is

$$\frac{4}{2} = \frac{8}{4} = \frac{16}{8} = \frac{32}{16} = \ldots\ldots\ldots\ldots\ldots\ldots$$

So the common ratio in this case is (r) =2

Note that like in arithmetic progression (A.P), in geometric progression (G.P) the common ratio does not change.

Also the terms may be decreasing instead of increasing, the geometric sequence or series whose terms decrease have a positive common ratio which is less than 1 for the progression with positive terms.

Eg. In the sequence 8, 4, 2, 1…. The common ratio is r = $\frac{4}{8}$ or $\frac{2}{4}$ or $\frac{1}{2}$

So the common ratio is $\frac{1}{2}$.

Generally if $G_1, G_2, G_3, G_4, \ldots\ldots G_n$ are the terms geometric sequence then the common ratio is

$$r = \frac{G_2}{G1} = \frac{G_3}{G2} = = \frac{G_4}{G3} = \frac{G_n}{Gn-1}$$

Example 9

For each of the following sequence find the common ratio.

(a) 3, 6, 12, 24, ……………………………….

(b) $1\frac{1}{2}, = \frac{1}{4}, = \frac{1}{8}$ ……………………………………

Solution;

(a) 3, 6,12, 24, ……………………………….

$$r = \frac{G_2}{G1} = \frac{G_3}{G2} = \frac{G_4}{G3}$$

$$r = \frac{6}{3} = \frac{12}{6} = \frac{24}{12} = 2$$

∴ The common ratio (r) = 2

(b) 1, $\frac{1}{2}$, $\frac{1}{4}$, $\frac{1}{8}$, ………………………………….

$$r = \frac{G_2}{G1} = \frac{G_3}{G2} = \frac{G_4}{G3}$$

$$r = \left(\frac{1}{2} \div 1\right) \text{or} \left(\frac{1}{4} \div \frac{1}{2}\right) \text{ or } \left(\frac{1}{8} \div \frac{1}{4}\right)$$

$$r = \frac{1}{2}$$

∴ The common ratio (r) = $\frac{1}{2}$

Example 10

For the following geometric sequences, find the common ratio and write down the next two terms:

(a) 2, 3, $4\frac{1}{2}$, $6\frac{3}{4}$..

(b) 10, -5, 2.5, -1.25 ....

Solution:

(a) Since 2,3, $6\frac{3}{4}$ ..... is a geometric sequence, it has a common ratio (r) which is found as follows

$$r = \frac{G_2}{G1} = \frac{G_3}{G2} = \frac{G_4}{G3}$$

$$r=\frac{3}{2}, = \frac{4\frac{1}{2}}{3} = \frac{6\frac{1}{2}}{4\frac{1}{2}}$$

$$r= \frac{3}{2}$$

$\therefore$ The common ratio (r) = $\frac{3}{2}$, or 1.5

The next term is found by multiplying the term considered to be the last term by the common ratio.

So the next term to $6\frac{3}{4}$ is $6\frac{3}{4} \times 1.5 = \frac{81}{8}$

and the next term to $\frac{81}{8}$ is $\frac{81}{8} \times 1.5 = \frac{243}{16}$,

$\therefore$ The next two terms of the sequence are $\frac{81}{8}$ and $\frac{243}{16}$,

(b) 10,-5, 2.5, -1.25, ……………………………………

$r = \frac{-5}{10} = \frac{2.5}{-5} = \frac{1.25}{2.5}$

$\therefore$ The common ratio (r) $= -\frac{1}{2}$

The next two terms are:

$\left(-1.25 \times (\frac{-1}{2})\right)$ and $\left(-1.25 \times (\frac{-1}{2})\right) \times \left(\frac{-1}{2}\right)$

Which are $\frac{5}{8}$, and $\frac{-5}{16}$

$\therefore$ The next two terms are $\frac{5}{8}$, and $\frac{-5}{16}$

Exercise 3

1. Which of the following sequences are geometric

a. 1, 2, 4, 8, 16, ……………………………………

b. 2, 6, 18, 54, 162, …………………………………

c. 1, -1,1,-1,1, ………………………………………

d. x^2, $2x^3$, 4x4, $8x^3$…………………………………

e. 1, 2, 4, 7, 10, ……………………………………

f. 0.1, 0.2, 0.3, 0.4, 0.5, ……………………………

g. 3, 6, 9, 12,15, ………………………………………

2. Find thecommon difference for each of the following geometric progressions (G.P)

(a) 1, -0.5, 0.25, -0.125, ………………………………

(b) $4x, 4x^2, 4x^3, 4x^4$, ………………………………

(c) $\frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \frac{1}{16}$,…………………………………………

(d) 1.1, 2.2, 4.4, 8.8, … … … . .

3. Find thenext term of the sequence 2, 10, 50, 500,………………….

4. The populationof a town is decreasing so that every year the population declines by a quarter. If the population is originally 100,000. What will it be after 5 years?

The General Term of an AP

Find the general term of an AP

If A_1, A_2, A_3, …………………A_n are the terms of an arithmetic sequence, then there is a common difference d which is given by

$d = A_2 – A_1 = A_3 – A_2 = A_n – A_n – 1$

that is $d = A_2 - A_1$ or

$d = A_3 - A_2$ or

$d = A_4 - A_3$ or

|

|

|

$d = A_n - A_{n-1}$

So $d = A_2 - A_1$, $d = A_3 - A_2$, ……………………………………………

Means $A_2 = A_1 + d$

$A_3 = A_2 + d$

But $A_2 = A_1 + d$

So, $A_3 = [A_1 + d] + d = A_1 + 2d$

But . $A_3 = A_1 + 2d$ which means

$A_4 = [A_1 + 2d] + d$

$= A_1 + 3d$

Putting into consideration this pattern, it is true that

$A_5 = A_1 + 4d$

$A_6 = A_1 + 5d$

$A_n = A_1 + (n-1)d$

Where A_n is the n^{th} term

The n^{th} term of the sequence with first term A_1 and common difference d is given by

$$A_n=A_1+(n-1)d$$

Example 11

Find theformula for the n^{th} term of the sequence 8 , 9.5, 11, 12.5, 14, 15.5,……

Solution

First term (A_1) = 8

Common difference (d) = A_2-A_1=A_3-A_2

Or A_6-A_5

d=9.5-8

or d=15.5-14

d=1.5

But $A_n=A_1+(n-1)d$

$A_n= 8+ (n-1) x1.5$

$A_n=8+1.5n-1.5$

$A_n=6.5+1.5n$

$\therefore A_n=1.5n+6.5$

Note that the n^{th} term gives every term in the sequence,

For example when n=3, you have A_3=1.5x3+6.5=11

So A_3=11 where 11 is given in the sequence above having the third position.

Therefore A_n shows the position of the term in sequence and of $A_1+(n-1)d$ gives the value of the term for any positive integer.

Example 12

The 5th term of an arithmetic sequence is 11, and the 8th term is 26. Find the first five terms.

Solution:

Given that

$A_5=11$ and $A_8 = 26$

From

$A_n=A_1+ (n-1)\ d$

$11=A_1+ (5-1)\ d$

$11=A_1+4d$ and

$26=A_1+ (8-1)\ d$

$26=A_1+7d$

So $A_1+4d=11$ …………….. (1)

$A_1+7d=26$…………………(2)

Solving for A_1 and d simultaneously gives

$d=5$ and $A_1=-9$

But $A_n=A1+ (n-1)\ d$

$A_n= -9+ (n-1)\ x5$

$A_n=-9+5n-5$

$A_n=5n-14$

$A_n=5n-14$

So n=1

$A_1 = 5x1-14=-9$

n = 2

$A_2 = 5x2-14=-4$

n = 3

$A_3 = 5x3-14=1$

n = 4

$A_4 = 5x4-14=16$

n = 5

$A_5 = 5x5-14=11$

∴ The first five terms are -9,-4, 1, 6, and 11.

Example 13

The 8th term of an arithmetic sequence is 9 greater than the 5th term, and the 10th term is 10 times the 2nd term. Find

a. The common difference (d)

b. 20th term.

Solution:

Let A_1, A_2, A_3,……………………… A_n

Be the terms of the given sequence A_8 > A5 by 9 means

$A_8 - A_5 = 9$ and

A_{10} is 10 times the second term means

$A_{10}=10A_2$

But from $A_n=A_1+ (n-1) d$,

$A_8=A_1+7d$

$A_5=A_1+4d$

$A_{10}=A_1+9d$

and $A_2=A_1+d$

So $A_1+7d-(A_1+4d) =9$

$3d=9$

$d=3$

Also, $A_{10}=10A_2$ means $A_1+9d=10(A_1+d)$

$A_1+9d=10A_1+10d$

$9A_1= -d$

$A_1= \frac{-d}{9}$

But d=3

$A_1= \frac{-3}{9}$

$A_1= \frac{-1}{3}$

Therefore the common difference (d) is 3,

To get the 20[th] term, use the n[th] term

I.e. $A_n=A_1+ (n-1)\, d$

$A_{20}=A_1+ (20-1)\, d$

$A_{20}= -\frac{1}{3} + 19 \times 3 = \frac{170}{3}$

$\therefore$ The 20[th] term is $\frac{170}{3}$

The General Term of GP

Find the general term of GP

If G_1, G_2, G_3,……………..G_n are the terms of a geometric sequence, then they have a common ratio (r) which is given by

$r = \frac{G_2}{G1} = \frac{G_3}{G2} = = \frac{G_4}{G3} = \frac{Gn}{Gn-1}$

This means $G_2=G_1r$, $G_3=G_2r$, $G_4=G_3r$ and $G_n=G_{n-1}r$

So $G_3= (G_1r)\ r$

Where $G_1r =G_2$

$G_3=G_1r^2$

$G_4= (G_1r^2)\ r$

Where $G_1r^2=G_3$

$G_4=G_1r^3$

Following this pattern, you find that $G_5=G_1r^4$, $G_6=G_1r^5$, $G_7=G_1r^6$ etc

We have seen that

$G_1 = G_1 = G_1 r^0$

$G_2 = G_1 r^1 = G_1 r$

$G_3 = G_1 r^2$

$G_4 = G_1 r^3$

$G_5 = G_1 r^4$

$G_6 = G_1 r^5$

|

|

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|

$G_n = G_1 r^{n-1}$

Where G_n is the n^{th} term.

$\therefore$ $$G_n = G_1 r^{n-1}$$

Example 14

Find the formula for the n^{th} term of each of the following geometric sequence.

a. 2, 6, 18, 54 , ………………………………

b. 4,-2, 1, -0.5, 0.25 …………………………

Solution:

(a) 2, 6, 18, 54 ,………………………………..

(b) From the sequence 2,6,18,54

(c) $G_1=2, G_2=6, G_3=18$ and $G_4=54$

The common ratio (r) $= \frac{G_2}{G1} = \frac{G_3}{G2}$

So r = $\frac{6}{2} = \frac{18}{6} = \frac{54}{18} = 3$

$r=3$

But $G_n = G_1 r^{n-1}$

$G_n = 2(3)^{n-1}$

$G_n = 2x\ (3^n)\ x\ (3)^{-1}$

$G_n = 2\times (3^n) \times \frac{1}{3}$

$G_n = \frac{2}{3} \times 3^n$

$\therefore$The n^{th} term is $G_n = \frac{2}{3} \times 3^n$ or $G_n = 2\times (3^{n-1})$

(c) 4, -2, 1, -0.5, 0.25,………………

G_1=4

G_2=-2

$\frac{G_2}{G1} = \frac{G_3}{G2} = \frac{-2}{4} = \frac{1}{-2}$ =r

r=$\frac{1}{-2}$

but $G_n= G_1r^{n-1}$

$G_n=4\times \left(\frac{-1}{2}\right)^{n-1}$

∴The nth term is $G_n = 4\times \left(\frac{-1}{2}\right)^{n-1}$

Example 15

Considering that,

A geometric sequence is given by 3, 1, $\frac{1}{3}$, $\frac{1}{9}$, …………………..

Find the 13th term of this sequence.

Solution;

Given the sequence3, 1, $\frac{1}{3}$, $\frac{1}{9}$, …………………..

$G_1=3$

$G_2=1$

$G_2=\frac{1}{3}$

$G_4=\frac{1}{9}$, $G_{13}=?$

$r=\frac{G_2}{G1}=\frac{G_3}{G2}=\frac{1}{3}=\frac{\frac{1}{3}}{1}$

$r=\frac{1}{3}$

from $G_n=G_1r^{n-1}$

G_{13} is found when n=13

So $G_{13}=3\left(\frac{1}{3}\right)^{13-1}$

$G_{13}=3\left(\frac{1}{3}\right)^{12}$

$=3\times\frac{1}{3}\times\left(\frac{1}{3}\right)^{11}$

$=\left(\frac{1}{3}\right)^{11}=3^{-11}$

$\therefore$ The 13^{th} term is $\left(\frac{1}{3}\right)^{11}$ or 3^{-11}

Exercise 4

1. In the arithmetic sequence, the 17^{th} term is 30 and 9^{th}term is 42 find the first three terms.

2. In the Arithmeticsequence the third term 12 and the 9^{th}term 24. Find the n^{th} term of the sequence and use it to find the 15^{th} term.

3. Find the15^{th} term of the sequence 5, 10, 20, 40 ,…………………………

4. A population isincreasing and every year it is multiplied by 1.03. If it starts off at 10,000,000, what will it be after n years?

5. The first termof the geometric sequence is 7 and the common ratio is 4. What is the 9th term of this sequence?

Series

The Formula for a Sum of an Arithmetic Progression

Derive the formula for a sum of an arithmetic progression

When the terms are separated by addition (+) sign, there we have what we call a series.

Example: 2+4+6+8+……………………………

Is a series with the first term (A_1) 2 and common difference (d) 2

It is possible to establish a formula for the sum of the first n terms of the arithmetic progression.

Let S_n denote the sum of the first n terms of the arithmetic series.

Consider the sum of the first 5, terms of arithmetic progression (AP) whose first term is 1 and whose common difference (d) is 1.

So $S_5 = A_1+A_2+A_3+A_4+A_5$

$S_5=1+2+3+4+5$ ………………………….. (1)

The first case is the sum of five terms which are increasing from 1 up to 5 while the second case shows the same sum but the terms are decreasing from 5 to 1.

If you add (1) and (2) together, you find that

$S_5+S_5=(1+5) + (2+4) +(3+3) + (4+2) + (5+1)$

$25_s = 6+6+6+6+6$

$25_5=30$

Dividing by 2 each side gives

$S_5 = \frac{30}{2} = 15$

$\therefore 1+2+3+4+5=15$

Now for any number of terms (n)

$S_n = A_1 + A_2 + A_3 + \ldots\ldots\ldots\ldots + A_{n-1} + A_n$

Which is the same as

$S_n = A_n + A_{n-1} + A_{n-2} + \ldots\ldots A_2 + A_1$

So $2S_n = (A_1+A_n) + (A_2+A_{n-1}) + (A_3+A_{n-2}) + (A_{n-1}+A_2) + (A_n+A_1)$

Also we can write

$S_n = A_1 + (A_1+d) + (A_1+2d) + \ldots\ldots\ldots\ldots\ldots\ldots + A_n$

And $S_n = A_n + (A_n-d) + (A_n-2d) + \ldots\ldots\ldots\ldots\ldots\ldots + A_1$

Which means

$2S_n = (A_1+A_n) + (A_1+d+A_n-d) + (A_1+2d+A_n-2d) + \ldots\ldots + (A_n+A_1)$

$2S_n = (A_1+A_n) + (A_1+A_n) + (A_1+A_n) + \ldots\ldots\ldots\ldots\ldots$ n times

$2S_n = n \times (A_1+A_n)$

$S_n = \frac{n}{2}(A_1+A_n)$

but $A_n = A_1 + (n-1)\, d$

$S_n = \frac{n}{2}(A1 + A1 + (n-1)d)$

$S_n = \frac{n}{2}\left(2A_1 + (n-1)\, d\right)$

$\therefore$The sum of n terms in arithmetic progression is

$$S_n = \frac{n}{2}\,[\,2A_1 + (n-1)\, d\,]$$

Example 16

Findthe sum of the first 20 terms of the series

$1+ \ 1\frac{1}{2} + 2\frac{1}{2} + \ \ldots\ldots\ldots\ldots$

Solution:

$S_n = \frac{n}{2}(2A_1 + (n-1)d\ )$

From the series above

$A_1=1$, $d=\frac{1}{2}$ and $n=20$

Therefore $S_{20} = \frac{20}{2}\left(2 \times 1 + (20-1)\frac{1}{2}\right)$

$S_{20} = 10\left(2 + \frac{19}{2}\right)$

$= 10 \times \frac{23}{2} = 115$

$\therefore S_{20} = 115$

Example 17

Find the sum of the series 4+7+10+13+…………….+304

Solution:

To use the formula for summation of n terms, you must know how many terms are there, i.e finding the value of n;

Now

$A_1=4$, $d=3$ and $A_n = 304$ $n=?$

$A_n = A_1 + (n-1)d$

$304 = 4 + (n-1)x3$

$304 = 4+3n-3$

$304=3n+1$

304=3n

n =$\frac{303}{3}$ = 101

Therefore we are required to find the sum of the 101 terms of the given series.

But $S_n=\frac{n}{2}((2A1+(n-1)d)$

S_{101} **=** $\frac{101}{2}(2\times 4+(101-1)x3)$

S_{101} = $\frac{101}{2}(8+300)$

= $\frac{101}{2}(308)$

S_{101} = 101×154=15,554

∴ The sum of 4+7+10+13+ …………………………+304 is **15,554.**

Example 18

How many terms of the series 1+3+5+7+………………. are needed to make the sum of 169?

Solution:

S_n=169

A_1=1

d=2

n=?

From $S_n = \frac{n}{2}(2A_1 + (n-1)d)$

$169 = \frac{n}{2}(2 \times 1 + (n-1) \times 2)$

$169 = \frac{n}{2} \times (2n)$

$169 = n^2$

$n = \sqrt{169} = 13$

∴13 terms are required.

Exercise 5

1. Find thesum of the first 20 terms of the series

a. 2+5+8+11+……………………

b. 19+16+13+10+7+……………

2. Find thenumber of terms and the sum of the series:

a. 1+3+5+7+ ……………………………………

b. 40+37+34+31+…………………+-257

3. The sumof the first 10 terms of an arithmetic progression (A.P) is 40, and the sum of the next 10 terms is 80. Find the sum of the first five terms of the series.

4. One dayFrola spends 40 minutes of her home work. The length of time she spends increase by 4 minutes each day. Find the total length of time she spends after eight days.

The Arithmetic Mean

Calculate the arithmetic mean

Remember that the arithmetic mean (M) of n numbers is found by adding them and then dividing the sum by n, e.g the arithmetic mean of a,b,c and d is

$$M = \frac{a+b+c+d}{4}$$

The Formula for the Sum of a Geometric Progression

Derive the formula for the sum of a geometric progression

Geometric series are the series that can be written as

$G_1+G_2+G_3+$ ………………………..G_n

Example: 2+4+8+16+ ………………………..$+G_n$

Or 1+3+9+27+81+…………………….

Suppose we want to find the sum of 1+3+9+9+27+81+…………………

S_5=!+3+9+27+81……………………….(1)

If we multiply s_n by the common ratio(r), we have.

$rS_5 = r(1 + 3 + 9 + 27 + 81)$

but r=3

So $3S_5 = 3+9+27+81+243$ ……………….(2)

Subtracting (1) from (2) gives

$3S_5 - S_5 = (3+9+27+81+243)-(1+3+9+27+81) = 243-1$

$2 S_5 = 242$

$S_5 = \frac{242}{2} = 121$

In general, for a series with terms $G_1+G_2+G_3+$ …………………………G_n, and common ratio($r \neq 1$),

$S_n = G_1+G_1r+G_1r^2+$…………………..$+G_1r^{n-1}$

$rS_n = G_1r+ G_1r^2+ G_1r^3+$……………….$+G_1r^n$

Now $rS_n - S_n = G_1r^n - G_1$

$rS_n - S_n = G_1 (r^n-1)$

$S_n(r - 1) = G_1 (r^n-1)$

$S_n = \frac{G_1(r^n-1)}{r-1}$

$\therefore$ $S_n = \frac{G_1(r^n-1)}{r-1}$

NB; If $-1 < r < 1$, it is easier to use the formula in the form of

$$S_n = \frac{G_1(1-r^n)}{1-r}$$

Example 19

1. Find the sum of the geometric series 2+4+8+ ………………..+2048

Solution:

$G_1=2$, $r=2$

$G_n=2048$, n?

$S_n=?$

From $G_n=G_1r^{n-1}$

$2048 = 2\times(2^{n-1})$

$2048= 2\times2^n\times \frac{1}{2}$

$2^{11}=2^n$

n=11

Also $S_n = G_1 \frac{(r^n-1)}{(r-1)}$

$S_{11} = 2 \times \frac{(2^{11}-1)}{(2-1)}$

$=2(2^{11}-1)$

$=2(2048-1)$

$=4094$

∴The sum is **4094**

Example 20

Find the sum of the first 8 terms of the series 5+20+80+320+ ……………

Solution

$G_1=5$, $r = \frac{G2}{G1} = \frac{20}{5} = 4$

$r=4$

$S_8=?$

From $S_n = \frac{G1(r^n-1)}{r-1}$

$S_8 = 5 \times \frac{(4^8-1)}{4-1}$

$S_8 = \frac{5\times(65{,}536-1)}{3}$

$= \frac{5\times 65{,}536}{3}$

$S_8 = 109{,}225$

∴The sum is **109,225**

Exercise 6

1. For each ofthe following series, find the number of terms and hence the sum of the series.

a. 1+3+9+…………………+729

b. 1-2+4-8+………………+1,024

2. Find the sum of the first 10 terms of the series

(a) 4+2+1+½ +………………………………..

(b) $4-2+1-\frac{1}{2}+$……………………………………

3. Masanja sets off on a long Journey. The first day he walks 30km, but the distance he walks each day is 10% less than on the previous day. Find the total distance he has walked after 12 days.

The Geometric Mean

Calculate the geometric mean

The Geometric mean (GM) of n positive numbers is found by taking the n^{th} root of their product.

Example 21

The Geometric mean of a, b, c and d is

$GM = \sqrt[4]{abcd}$

Therefore the arithmetic mean of 3 and 5 is

$$M=\frac{3+5}{2}=\frac{8}{2}=4$$

$\therefore M=4$

But the geometric mean of 3 and 5 is

$G.M\ \sqrt{3x5}$

$G.M=\sqrt{15}\ = 3.87$

$\therefore$The geometric mean of 3 and 5 is G.M=3.87

The arithmetic mean and geometric mean can be used to check that a sequence is an arithmetic or geometric respectively.

FACTS:

1. If a, b and c are consecutive three term of arithmetic progression (A.P), then b is the arithmetic mean of a and c

2. If a, b and c are three consecutive terms of geometric progression (G.P), then b is the geometric mean (G.M) of a and c.

Proof:

(1) Suppose a, b and c are the three consecutive terms of an arithmetic progression then

d=b-a or d=c-b

So. b-a=c-b

b+b=a+c

2b=a+c

b=$\frac{a+c}{2}$ as required.

(ii) Let a, b and c be the consecutive terms of geometric progression (G.P)

Then r=$\frac{b}{a}$ or r = $\frac{c}{b}$

So $\frac{b}{a} = \frac{c}{b}$

bxb=axc

b^2=axc

b=$\sqrt{axc}$ as required.

Example 22

Find the arithmetic and geometric means of

(a) 3 and 12 (b) 5,8,14

Solution

(a) Arithmetic mean

$M = \frac{3+12}{2} = \frac{15}{2} = 7.5$

M=7.5

$G.M = \sqrt{3 \times 12} = \sqrt{36} = 6$

G.M=6

∴The arithmetic and Geometric means of 3 and 12 are 7.5 and 6 respectively.

(b) $M = \frac{5+8+14}{2} = \frac{27}{3} = 9$

$G.M = 3\sqrt{5x8x14} = 3\sqrt{560} = 8.24$

∴The arithmetic and geometric mean of 5, 8 and 14 are 9 and 8.24 respectively.

Exercise 7

1. Find the arithmetic and geometric means of the following;

a. x_1, x_3

b. 4x, 9x

c. 4a, 25a.

2. The arithmeticmean and geometric mean of two numbers are 7.5 and 6 respectively. Find the two numbers.

Compound Interest

Compound Interest using Formula

Calculate compound interest using formula

Suppose money is invested or borrowed. At the end of a year, interest is calculated. Suppose this interest is added to the original principal, and at the end of the next year interest is added to the new principal. This process may be continued for a number of years.

This process is called *COMPOUND INTEREST.*

When money is invested at a compound interest, the amount of money increase as a geometric sequence.

Example 23

Ibrahimu invested 20,000/= at 6% compound interest. How much was there after 5 years?

Solution:

Increasing by 6% is equivalent to multiplying by $(1+\frac{6}{100}) = 1.06$

So after one year the amount is

$20,000 \times 1.06$

After two years: $20,000 \times 1.06 \times 1.06$

$= 20,000 \times (1.06)^2$

After 3 years: $20,000 \times (1.06)^3$

After 4 years: $20,000 \times (1.06)^4$

After 5 years: $20,000 \times (1.06)^5$

$\approx 26,765.$

$\therefore$ After 5 years, the amount will approximately be 26,765/=

Now let the principal be P, the rate R% and the time in years be n.

The interest in one year is $\frac{PR}{100}$

The amount after one year is

$$P + \frac{PR}{100} = P(1 + \frac{R}{100})$$

So the new principal is $P(1 + \frac{R}{100})$

So the interest in the second year is $\frac{PR}{100} = P\left(1 + \frac{R}{100}\right)R$

So after two years the principal is $= P\left(1 + \frac{R}{100}\right)\left(1 + \frac{R}{100}\right)$

Which is $P\left(1 + \frac{R}{100}\right)^2$ and so on.

Therefore n years the principal is $A = P\left(1 + \frac{R}{100}\right)^n$

Example 24

At the beginning of each year Martha invests 10,000/= at 5% compound interest. How much does she have at the end of the 10^{th} year?

Solution:

She has made 10 different investments each giving different amount of interest.

The 1^{st} investment has had 10 years of interest, hence it is $10{,}000 \times (1.05)^{10}$

The 2^{nd} investment has had 9years of interest. So it is $10{,}000 \times (1.05)^{9}$

The 3^{rd} investment has had 8 years of interest. Hence it is $10{,}000 \times (1.05)^{8}$

Following this pattern,

The 10^{th} investment has had 1 year of interest. Hence it is 10,000x1.05.

The sum of all these amounts is given by;

$10{,}000\times1.05^{10} + 10{,}000 \times 1.05^{9} + 10{,}000\times1.05^{8} + \ldots\ldots\ldots\ldots + 10{,}000\times1.05$

This geometric series with first term 10,000 x 1.05 and common ratio 1.05.

Hence the sum is $\frac{10,500(1.05^{10}-1)}{1.05-1}$ = 132,068.

∴She has 132,000/= (nearest 100/-)

Exercise 8

1. Find the total amount of the following savings if they earn compound interest.

a. 100,000/= for 2 years at 6% p.a

b. 250,000/= for 3 years at 4.5% p.a

c. 400,000/= for 20 years at 5.5% p.a

2. A population is increasing at 2% if it starts at 10,000,000 what will it be after 20 years.

3. At the beginning of each year 600,000/= is invested at 6% compound interest.Find the total value of the investment at the end of the 15^{th} year.